Philosophy 2310
Definitions and Terms used

A note to the reader: There are many different ways of defining some of these terms which in the end turn out equivalent. In all cases the definition I give agrees with the book's definition if it has one. LPL has a helpful glossary at the end of the book. But in reading elsewhere you may come across different definitions for some of these terms.

Throughout, Γ refers to an arbitrary set of sentences such as $\{P, Q, R\} = \Gamma$ or $\{\forall xP(x), \forall xQx, \exists x\forall yRxy\} = \Gamma$

φ and ψ refer to arbitrary sentences such as when $P \vdash Q$ might be represented as $\varphi \vdash \psi$ or the conjunction $\varphi \wedge \psi$ might represent any of $P\wedge Q$, $\forall xP(x) \wedge \forall xQ(x)$, or $\exists xP(x) \wedge (P(a) \vee Q(b))$

=def means that what follows is the definition of what came before (is equal by definition to…) Example: The definition of a tautology is a sentence which is true in every truth value assignment. It is true that all tautologies are theorems, but it is not true *by definition* that all tautologies are theorems. All tautologies are theorems because of the Completeness Theorem.

Proof terms:

$\Gamma \vdash \varphi$ =def there is a proof of φ (φ is the last line) where all of the premises are members of Γ (there might be more members of Γ than are used in the proof). To count as a proof, each line has to be justified by some basic rule of our proof system $\mathcal{F}$

Example: $\{P\rightarrow Q, P\} \vdash Q$

Γ is proof-inconsistent =def $\Gamma \vdash \perp$
Γ is proof-consistent =def Γ is not proof-inconsistent.

Examples: $\{P\&Q, Q\rightarrow R\}$ is proof-consistent. $\{P, Q, \neg P\vee\neg Q\}$ is proof-inconsistent.

φ is a theorem =def $\vdash\varphi$ (that is, $\Gamma\vdash\varphi$ when Γ=the empty set which has no members – so φ is provable from no premises at all.)

Example: $\vdash P\vee\neg P$

φ is stronger than ψ =def $\varphi\vdash\psi$ and it is false that $\psi\vdash\varphi$.
φ is weaker than ψ =def ψ is stronger than φ.
φ is proof-equivalent to ψ =def $\varphi\vdash\psi$ and $\psi\vdash\varphi$

Examples: P is stronger than $P \vee Q$
Q is weaker than $R \wedge Q$
$\neg(P \vee Q)$ is proof-equivalent to $\neg P \wedge \neg Q$

Semantic terms:

$\Gamma \models \varphi$ means that φ is a consequence of Γ. It also means that Γ entails φ.
$\Gamma \models \varphi$ =def every interpretation (TVA in SL) that makes every sentence in Γ true also makes φ true. This is equivalent to saying that there is no interpretation (TVA in SL) that makes all the members of Γ true and also makes φ false.

Example: $\{\forall xP(x), \forall x(P(x) \to Q(x))\} \models \forall xQ(x)$

Γ is consistent =def there is an interpretation in which all of the members of Γ are true.
Γ is inconsistent =def Γ is not consistent.
An interpretation which makes all of the members of Γ true is called a Model of Γ.

Examples: $\{\exists xP(x), \exists x\neg P(x)\}$ is consistent.
$\{\forall xP(x), \forall x\neg P(x)\}$ is inconsistent.
I = D:{a} is a model of $\{\exists x\ x{=}x\}$

[[[Note that in almost all advanced texts, "consistency" is a proof-theoretic term while "satisfiability" is the semantic term.]]]

φ is a tautology =def φ is true in every truth-value assignment
φ is an FO validity =def φ is true in every interpretation.
φ is a contradiction =def there is no interpretation in which φ is true.
φ is contingent =def φ is neither valid nor a contradiction. (This is equivalent to saying that there is an interpretation in which φ is true and at least one in which φ is false.)

Examples: $P \vee \neg P$ is a tautology
$\exists x(P(x) \vee \neg P(x))$ is valid
$\exists xP(x)$ is contingent
$\exists x(P(x) \wedge \neg P(x))$ is a contradiction

φ is independent of ψ =def $\{\varphi, \psi\}$ and $\{\neg\varphi, \psi\}$ are both consistent.
φ and ψ are independent (of each other) =def all three of $\{\varphi, \psi\}$, $\{\varphi, \neg\psi\}$,$\{\neg\varphi, \psi\}$, are consistent.
φ is independent of Γ =def $\Gamma \cup \varphi$ (Γ added together with φ) and $\Gamma \cup \neg\varphi$ are both consistent.
Γ is mutually independent =def there is no $\Gamma_1 \subset \Gamma$ (no proper subset of Γ) such that $\Gamma_1 \models \varphi$ for some φ in Γ and φ not in Γ_1. In other words, if you take away some sentence φ from Γ then the rest of the sentences don't entail φ.

Examples: $\exists xP(x)$ and $\exists xQ(x)$ are independent.
$\exists xP(x)$ is independent of $\{\forall x(P(x) \to Q(x)), \forall x(Q(x) \to R(x))\}$
$\{\forall xR(x,x), \forall x\forall y(R(x,y)\to R(y,x)),$
$\forall x\forall y\forall z[((R(x,y) \wedge R(y,z)) \to Rxz)\}$ is mutually independent.

Metatheorems that relate proofs and semantics:

The Soundness Theorem says that for any Γ and for any φ, if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$. This is equivalent to: if $\Gamma \cup \neg\varphi$ is proof-inconsistent then $\Gamma \cup \neg\varphi$ is inconsistent. This is equivalent (by contraposition) to if $\Gamma \cup \neg\varphi$ is consistent then $\Gamma \cup \neg\varphi$ is proof-consistent. Since these are arbitrary this is equivalent to: for any Γ, if Γ is consistent then Γ is proof-consistent.

The Completeness Theorem is the converse of the Soundness Theorem. It says that for any Γ and any φ, if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

Other major metatheorems of FOL:

Compactness Theorem: Γ is consistent if and only if every finite subset of Γ is consistent.

Church's Theorem: First order logic is undecidable (as long as the language contains at least one two-place predicate). This means that there is no algorithm for correctly answering yes or no to the following question: is $\Gamma \models \varphi$ true or false? (this algorithm would have to work for any Γ, φ) Equivalently, there is no algorithm for answering: is Γ consistent? Is φ a theorem?, etc.

For the final exam

For our final exam, in addition to the terms above, you should also know and be able to manipulate the following properties of relations:

R is reflexive =def R satisfies the condition that $\forall x\, R(x,x)$
R is symmetric =def R satisfies the condition that $\forall x \forall y[R(x,y) \rightarrow R(y,x)]$
R is transitive =def R satisfies the condition that $\forall x \forall y \forall z[((R(x,y) \wedge R(y,z)) \rightarrow Rxz)]$